Introduction to Polyhedral Combinatorics

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1 Introduction to linear programming

Basic notation

The **scalar product** of vectors $x, y \in \mathbb{R}^n$ is $xy := \sum_{i=1}^n x_i y_i$. We say that x is **orthogonal** to y if $xy = 0$. A (closed) **half-space** defined by $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ is the set $\{x \in \mathbb{R}^n : ax \le \alpha\}$. Note that $\varnothing$, and $\mathbb{R}^n$ are half-spaces according to our definition. A half-space is called homogeneous if $\alpha = 0$. Sometimes we consider vectors as row-vectors other times as column-vectors, it will be clear from the context when it counts. The intersection of finitely many (at least one) half-spaces is called a (convex) **polyhedron**, formally $\boldsymbol{P(A,b)} := \{x \in \mathbb{R}^n : Ax \le b\}$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ and the inequality meant coordinate-wise. A **linear programming problem** is the following. There is a polyhedron $P \subseteq \mathbb{R}^n$ and a $c \in \mathbb{R}^n$ and we want to maximize the scalar product cx where $x \in P$. Here c is called a **linear objective function**. It may happen that $P = \varnothing$ or cx is unbounded on P. A **polyhedral cone** is the intersection of finitely many homogeneous half-spaces: $\boldsymbol{P(A)} := \{x \in \mathbb{R}^n : Ax \le \underline{0}\}$.

Fourier-Motzkin elimination

The **outer projection** of a $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ to the last n coordinates is $(x_1, \dots, x_n) \in \mathbb{R}^n$. (The inner projection to the last n coordinates is $(0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$.) Projection for any subset of coordinate can be defined similarly.

1.1 Theorem (Fourier-Motzkin elimination)**.** *The outer projection of a polyhedral cone to some of its coordinates is an polyhedral cone.*

Proof. Suppose the polyhedral cone lives in $\mathbb{R}^{n+1}$. It is enough to show that the outer projection to all but one coordinate is an polyhedral cone since then we can repeat the process. By symmetry it is enough to handle the projection to the last n coordinates.

We need to prove that for every $A \in \mathbb{R}^{m\times(n+1)}$ there is a $A' \in \mathbb{R}^{m'\times n}$ such that $P(A') = \{x \in \mathbb{R}^n | \exists\alpha \in \mathbb{R} : (\alpha, x) \in P(A)\}$. We write $\boldsymbol{a_{i.}}$ for the i-th row of A and $\boldsymbol{a_{.i}}$ for the i-th column. Multiplying a row of A by a positive scalar does not change $P(A)$, hence we may assume that $a_{.1}$ does not have any other coordinates than $0, 1, -1$. If a row of A starts with 0, then put this row to A' without the initial 0 coordinate. Whenever $(1, p)$ and $(-1, q)$ are rows of A we put the row $p + q$ to A'. The definition of A' is complete (the order of rows obviously does not matter). From the construction is clear that if $(\alpha, x) \in P(A)$, then $x \in P(A')$.

Suppose that $x \in P(A')$. We need to find an $\alpha \in \mathbb{R}$ such that $(\alpha, x) \in P(A)$. If a row of A starts with 0, than the corresponding inequality will hold for (α, x) whatever α we choose. Rows of A of the form $(1, p)$ demand $\alpha \le -px$, the rows $(-1, q)$ require $qx \le \alpha$. Assume that $-px$ is the smallest upper bound and qx the largest lower bound that we obtain this way. Then all the requirements for α are $qx \le \alpha \le -px$. It is satisfiable if and only if $qx \le -px$ i.e. $(p + q)x \le \underline{0}$. But it holds, since $p + q$ is a row of A' and $A'x \le \underline{0}$ since $x \in P(A')$. □

1.2 Observation. It is worth to mention that if $\mathbb{F}$ is some ordered subfield of $\mathbb{R}$ (for example $\mathbb{Q}$) and in Theorem 1.1 the polyhedral cone is representable over $\mathbb{F}$, then the projection is also representable over $\mathbb{F}$. The analogue of this observation remains true for basically all of the further theorems as well.

1.3 Corollary. *The outer projection of a polyhedron to some of its coordinates is a polyhedron.*

Proof. It is enough to show that if $P := P(A, b) \subseteq \mathbb{R}^{n+1}$ is a polyhedron, then its projection $P' := \{x \in \mathbb{R}^n \,|\, \exists\alpha \in \mathbb{R} : (\alpha, x) \in P\}$ as well. Let $[Ab]$ be the matrix we obtain from A by adding b as a last column. Applying Theorem 1.1 take a matrix $[A'b']$ such that $P([A'b']) = \{x \in \mathbb{R}^{n+1} \,|\, \exists\alpha \in \mathbb{R} : (\alpha, x) \in P([Ab])\}$. We show $P' = P(A', b')$. On the one hand,

$$(\alpha, x) \in P \iff (\alpha, x, -1) \in P([Ab]) \implies (x, -1) \in P([A'b']) \iff x \in P'.$$

On the other hand,

$$(x, -1) \in P([A'b']) \implies \exists\alpha : (\alpha, x, -1) \in P([Ab]).$$

□

Polyhedral cones and generated cones

A **generated cone** consists of the non-negative combinations of finitely many (at least one) vectors: $\{\sum_{i=1}^m \lambda_i v_i : \lambda_i \in \mathbb{R}^+\}$ where $v_1, \dots, v_m \in \mathbb{R}^n$ are the generator vectors. The cone generated by the rows of the matrix Q is denoted by $\boldsymbol{G(Q)}$ i.e., $G(Q) = \{yQ : y \ge \underline{0}\}$.

1.4 Remark. If $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then $ax = \alpha$ can be expressed by two linear inequalities: $ax \le \alpha$, $-ax \le -\alpha$. Thus we may allow equalities in the description of polyhedrons.

1.5 Claim. *Every generated cone is an polyhedral cone.*

Proof. Clearly $G(Q) := \{z \mid \exists x \geq \underline{0} : xQ = z\}$. The system $\{(x,z) : xQ - z = 0, x \geq \underline{0}\}$ is a polyhedral cone. On the one hand, by Theorem 1.1, its outer projection to the z-coordinates is a polyhedral cone as well. On the other hand, it is exactly $G(Q)$. □

1.6 Lemma (Farkas lemma)**.** *$\nexists x \geq \underline{0} : Ax = b$ if and only if $\exists y : yA \leq \underline{0}$, $yb > 0$.*

Proof. First we show that solutions x, y for the two systems cannot simultaneously exist. Suppose that x, y are solutions. Then by multiplying the equation $Ax = b$ from left by the row vector y we obtain $(yA)x = yb$. Here the right side is positive by assumption but the left side is not since $x \geq \underline{0}$ and $yA \leq \underline{0}$ which is a contradiction. Suppose that $\nexists x \geq \underline{0} : Ax = b$. It means that the cone G generated by the columns of A does not contain b. By Claim 1.5 there is a matrix C such that $G = \{x : Cx \leq \underline{0}\}$. Then for an appropriate row y of C we have $yb > 0$ but for every column a of A we know $ya \leq 0$. □

1.7 Corollary (Fredholm alternative theorem)**.** *$\nexists x : Ax = b$ if and only if $\exists y : yA = \underline{0}, yb \neq 0$.*

1.8 Excercise (Farkas lemma general form). Show that exactly one of the following two systems are solvable.

$$\begin{aligned} &x_0 \geq \underline{0} \\ &Ax_0 + Bx_1 = b_0 \\ &Cx_0 + Dx_1 \leq b_1 \end{aligned}$$

$$\begin{aligned} &y_1 \geq \underline{0} \\ &y_0A + y_1C \geq \underline{0} \\ &y_0B + y_1D = \underline{0} \\ &y_0b_0 + y_1b_1 < 0 \end{aligned}$$

1.9 Claim. *If $P(A) = G(B)$, then $G(A) = P(B)$.*

Proof. From the conditions it follows that if a is a row of A and b is a row of B, then $ab \leq 0$ from which we obtain $G(A) \subseteq P(B)$. Suppose, to the contrary, that $x \in P(B) \setminus G(A)$, by Farkas-lemma 1.6 there is an y such that $ay \leq 0$ for every row a of A (thus $y \in P(A)$) but $xy > 0$. Since $P(A) = G(B)$ by assumption, y is a non-negative combination of the rows of B. Then $x \in P(B)$ implies $xy \leq 0$, which is a contradiction. □

1.10 Corollary. *Every polyhedral cone is a generated cone.*

Proof. Let $P(A)$ be given. By Claim 1.5 there is a matrix B such that $G(A) = P(B)$. But then by Claim 1.9 $P(A) = G(B)$. □

Bounds for a linear program

In this subsection let $P = \{x : Qx \leq b\}$ be a fixed nonempty polyhedron.

1.11 Theorem. *The following are equivalent:*

1. *$\{cx : x \in P\}$ is bounded from above,*
2. *$\nexists z\ Qz \leq \underline{0},\ cz > 0$,*
3. *$c \in G(Q)$.*

Proof. $1 \Longrightarrow 2$: if we have a z for which $Qz \leq \underline{0}$ and $cz > 0$, then for $x \in P$ and $\lambda \in \mathbb{R}^+$ we have $x + \lambda z \in P$ and

$$c(x + \lambda z) = cx + \lambda cz \to \infty \text{ if } \lambda \to \infty.$$

$2 \Longrightarrow 3$ follows from Farkas lemma 1.6.
$3 \Longrightarrow 1$: suppose that $c \in G(Q)$, i.e., there is some $y \geq \underline{0}$ for which $yQ = c$. Let $x \in P$ be arbitrary. Then

$$cx = (yQ)x = y(Qx) \leq yb.$$

□

1.12 Corollary (Weak duality)**.** *If $\{cx : x \in P\}$ is bounded from above, then $\{yb : yQ = c,\ y \geq \underline{0}\}$ is nonempty and any element of it is an upper bound of $\{cx : x \in P\}$.*

The system $\min yb,\ yQ = c,\ y \geq \underline{0}$ is the **dual** of the system $\max cx,\ Qx \leq b$.

1.13 Observation. The following sets are the same, it is called the **direction cone $\boldsymbol{D(P)}$** of P.

1. $\{z \mid \forall x \in P\ \forall \lambda \in \mathbb{R}^+\ x + \lambda z \in P\}$,
2. $\{z \mid \exists x \in P\ \forall \lambda \in \mathbb{R}^+\ x + \lambda z \in P\}$,
3. $\{z \mid Qz \leq \underline{0}\}$.

1.14 Observation. The following sets are the same, it is called the **translation space $\boldsymbol{T(P)}$** of P.

1. $\{z \mid \forall x \in P\ \forall \lambda \in \mathbb{R}\ x + \lambda z \in P\}$,
2. $\{z \mid \exists x \in P\ \forall \lambda \in \mathbb{R}\ x + \lambda z \in P\}$,
3. $\{z \mid Qz = \underline{0}\}$.

Basic solutions

Consider the system $Qx \leq b$ and let z be a solution. A row $q_{i.}$ of Q is called **$\boldsymbol{z}$-active** if $q_{i.}z = b_i$. Let the matrix $\boldsymbol{Q_z^=}$ consists of the z-active rows of Q and let $\boldsymbol{Q_z^<}$ be the rest.

1.15 Claim. *If $P = \{x : Qx \leq b\} = \{x : Q'x \leq b'\}$ and $z \in P$, then the row space of $Q_z^=$ and $Q_z'^=$ are the same. Furthermore, the row space of Q and Q' are the same as well.*

Proof. Suppose, to the contrary, that the rows of $Q_z^=$ do not span some row q of $Q_z'^=$. By Fredolm alternative theorem 1.7, there is some y such that $Q_z^= y = \underline{0}$ and $qy \neq 0$. , We may assume $qy > 0$ (by negating y if it is necessary). But then for a small enough $\varepsilon > 0$ we have $z + \varepsilon y \in \{x : Qx \leq b\}$ and $z + \varepsilon y \notin \{x : Q'x \leq b'\}$ which is a contradiction.

Similarly if for a row q of Q' is not spanned by the rows of Q, then there is an y such that $Qy = \underline{0}$ and $qy > 0$. But then for a large enough $\lambda > 0$ we have $z + \lambda y \notin \{x : Q'x \leq b'\}$ but $z + \lambda y \in \{x : Qx \leq b\}$. □

Let $P = \{x : Qx \leq b\}$ and $z \in P$. The **rank $\boldsymbol{r_P(z)}$** of z in P is $r(Q) - r(Q_z^=)$. By Claim 1.15, the rank depends just on the polyhedron and not on the inequality system that represents it. A $z \in P$ is a **basic solution** if $r_P(z) = 0$.

1.16 Theorem. *Let P be a nonempty polyhedron and let cx bounded from above on P. Then for all $x \in P$ there is a basic solution $x^* \in P$ for which $cx^* \geq cx$.*

Proof. We use induction on $r_P(x)$. If $r_P(x) = 0$, then $x^* := x$ is appropriate. Suppose we know the statement if the rank of x is at most n and let $r_P(x) = n + 1$. Assume that $P = P(Q, b)$. Since x is not a basic solution, the rows of $Q_x^=$ does not span all the rows of Q. By Fredholm alternative theorem 1.7, we have a z such that $Q_x^= z = 0$ but $qz \neq 0$ for some row q of Q. By negating z if necessary we can assume that $cz \geq 0$. If $cz > 0$, then there is some row p of Q for which $pz > 0$ otherwise $x + \lambda z \in P$ for $\lambda \in \mathbb{R}^+$ and $c(x + \lambda z) \to \infty$ if $\lambda \to \infty$. If $cz = 0$ then we can assume by negating z that $qz > 0$. Anyway, $cz \geq 0$ and $pz > 0$ for some row p of Q. Note that all such a row p is linearly independent from the rows of $Q_x^=$. Then for an appropriate $\lambda > 0$ we have:

1. $w := x + \lambda z \in P$,
2. all the x-active rows are w-active,
3. there is a w-active row which is linearly independent from the rows of $Q_x^=$.

Since $r_P(w) < r_P(x)$, by the induction hypothesis we have a basic solution x^* with $cx^* \geq cw$. But then using $cz \geq 0$ and $\lambda > 0$:

$$cx^* \geq cw = c(x + \lambda z) = cx + \lambda cz \geq cx$$

1.17 Remark. The "appropriate" λ is

$$\lambda := \min_{i : q_{i.}z > 0} \frac{b_i - q_{i.}x}{q_{i.}z}.$$

□

1.18 Corollary. *Every nonempty polyhedron has a basic solution.*

A basic solution x^* is a **strong basic solution** if the columns of Q corresponding to the non-zero coordinates of x^* are linearly independent i.e., $T(P)$ does not contain a nontrivial vector which has non-zero coordinates at most where x^* has non-zero coordinates.

1.19 Theorem. *Let P be a nonempty polyhedron and let cx bounded from above on P. Then for all $x \in P$ there is a strong basic solution $x^* \in P$ for which $cx^* \geq cx$.*

Proof. Let $x \in P := P(Q, b)$ be arbitrary. By Theorem 1.16, we may pick a basic solution x^* such that $cx^* \geq cx$ and x^* has a minimal number of non-zero coordinates among these. Suppose for a contradiction that x^* is not a strong basic solution and let z be a non-zero vector for which $Qz = \underline{0}$, and z has non-zero coordinates at most where x^*. Clearly $cz = 0$, otherwise cx would not be bounded on P. But then for a suitable $\lambda \in \mathbb{R}$ the vector $w := x^* + \lambda z$:

1. has more 0 coordinates than x^*,
2. is a basic solution since $Q^=_w = Q^=_{x^*}$,
3. $cw = cx^*$,

which contradicts the choice of x^*. □

When Q' is a submatrix of Q, then $\boldsymbol{b_{Q'}}$ stands for the outer projection of b to the coordinates corresponding those rows of Q which are in Q'.

1.20 Remark. By choosing maximally many independent rows and maximally many independent columns, the corresponding submatrix is quadratic and regular.

1.21 Claim. *For every strong basic solution x^*, there is regular submatrix Q' of Q of size $r(Q) \times r(Q)$ such that one can get x^* by taking the unique solution of $Q'x = b_{Q'}$ and extend it by 0 coordinates at those columns of Q which are not in Q'.*

Proof. Since x^* is a basic solution, the rows of $Q^=_{x^*}$ form a generator system of the row space of Q thus we can trim it to a base. The columns of Q corresponding to the non-zero coordinates x^* are linearly independent thus we can extend it to a base of the column space. Consider the submatrix Q' defined by these rows and columns.

It is routine to check that Q' satisfies the conditions. □

From Theorem 1.19 and Claim 1.21 we conclude the following.

1.22 Corollary. *A nonempty polyhedron has finitely many (but at least one) strong basic solutions.*

An $x_0 \in P$ is called an **optimal** solution (with respect to P and c) if $cx_0 \geq cx$ for every $x \in P$.

1.23 Theorem. *Let P be a nonempty polyhedron and let cx bounded from above on P, then there is a strong basic solution x^* which is optimal.*

Proof. By Theorem 1.19, for every solution there is a strong basic solution which is better. Claim 1.22 ensures that there are just finitely many strong basic solution thus we simply pick the best among these. □

Strong duality

1.24 Claim. *Let $P = P(Q, b)$ be nonempty and let c be an objective function. The following are equivalent.*

1. *$cx^* \geq cx$ for $x \in P$,*
2. *$\nexists z$ for which $Q^=_{x^*} z \leq \underline{0}$ but $cz > 0$,*
3. *$c \in G(Q^=_{x^*})$.*

Proof. $1 \Longrightarrow 2$: if such a z exists, then for a small enough $\varepsilon > 0$ we have $x^* + \varepsilon z \in P$ and it would be a better solution than x^*.

$2 \Longrightarrow 3$: it follows directly from Farkas lemma 1.6.

$3 \Longrightarrow 1$: according to 3, we have a $y^* \geq \underline{0}$ such that $y^*Q = c$ and y^* has non-zero coordinates at most at the rows $Q^=_{x^*}$. On the one hand, by Corollary 1.12 y^*b is an upper bound for $\{cx : x \in P\}$. On the other hand,

$$cx^* = (y^*Q)x^* = y^*(Qx^*) = y^*b,$$

where the last equation follows from the fact that y^* can have non-zero coordinates at most at the x^*-active rows. □

1.25 Corollary (Strong duality theorem of linear programming)**.** *If $P(Q, b) \neq \varnothing$ and cx is bounded from above on it, then* $\max\{cx : Qx \leq b\} = \min\{yb : yQ = c,\ y \geq \underline{0}\}$.

1.26 Corollary (Optimality criteria)**.** *x^* is an optimal **primal solution** (i.e., optimal with respect to $P(Q, b)$ and c) and y^* is an optimal **dual solution** (i.e., $y^* \geq \underline{0}$, $y^*Q = c$ and minimize y^*b among these) if and only if they are solutions and $y^*_i > 0 \Longrightarrow (q_{i.}x^* = b_i)$ holds for every i.*

1.27 Excercise (Strong duality theorem general form). Suppose that the linear program

$$\begin{aligned} &\max c_0x_0 + c_1x_1 \\ &x_0 \geq \underline{0} \\ &Ax_0 + Bx_1 = b_0 \\ &Cx_0 + Dx_1 \leq b_1 \end{aligned}$$

is bounded from above. Show that the optimum equals to

$$\begin{aligned} &\min y_0b_0 + y_1b_1 \\ &y_1 \geq \underline{0} \\ &y_0A + y_1C \geq c_0 \\ &y_0B + y_1D = c_1. \end{aligned}$$

The structure of polyhedrons

Let P be a nonempty polyhedron, cx bounded from above on P and $\delta = \max\{cx : x \in P\}$ (exists by Theorem 1.23). The set $F := \{x \in P : cx = \delta\}$ is called a **face** of P. Note that P itself is a face ensured by $c = \underline{0}$. **Proper faces** are the faces with $F \subsetneq P$.

1.28 Theorem. *Let $P = P(Q, b)$ be nonempty. Then a nonempty $F \subseteq P$ is a face of P if and only if we can partition the rows of Q into matrices Q', Q'' such that $\{x : Q'x \leq b_{Q'}, Q''x = b_{Q''}\} = F$.*

Proof. Let $F := \{x : Q'x \leq b_{Q'}, Q''x = b_{Q''}\} \neq \varnothing$ and let c be the sum of the rows of Q''. Then being an element of P which optimal for c means being in F.

Take a c for which cx is bounded on P. Pick an optimal dual solution y^* (see Corollary 1.26). Let Q'' consists of those $q_{i.}$ for which $y_i^* > 0$ and let Q' consist of the rest. Then by Corollary 1.26, an x^* is optimal if and only if the rows in Q'' are x^*-active i.e., if $x^* \in \{x : Q'x \leq b_{Q'}, Q''x = b_{Q''}\}$. □

1.29 Corollary. *A face of a face of a polyhedron P is a face of P.*

A linear combination $\sum_{i=1}^m \lambda_i v_i$ is **affine combination** if $\sum_{i=1}^m \lambda_i = 1$. A nonempty subset of $\mathbb{R}^n$ is an **affine subspace** if it is closed under affine combination. An affine combination is a **convex** combination if it is non-negative.

1.30 Observation. The affine subspaces are exactly the nonempty sets of the form $\{x : Ax = b\}$.

1.31 Claim. *For a $\subseteq$-minimal face F of the nonempty polyhedron $P(Q, b)$, there is a Q'' consisting of some rows of Q such that $F = \{x : Q''x = b_{Q''}\}$. Hence F is an affin subspace.*

Proof. Take a representation $F = \{x : Q'x \leq b_{Q'}, Q''x = b_{Q''}\}$ as in Theorem 1.28 where Q' has a minimal number of rows. If the Q' part is empty we are done. Otherwise consider a row $q'_{i.}$ of Q' as an objective function. Then $q'_{i.}x$ is bounded from above by $b_{Q'}(i)$. Let $\delta_i := \max\{q'_{i.}x : x \in F\}$. The set $\{x \in F : q'_{i.}x = \delta_i\}$ must be the whole F since F cannot have a proper face. It implies that $\delta < b_{Q'}(i)$ otherwise we can replace the corresponding inequality by equation. We show that

$$\{x : Q'x \leq b_{Q'}, Q''x = b_{Q''}\} = \{x : Q''x = b_{Q''}\}.$$

Assume for contradiction that $Q''z = b_{Q''}$ but $Q'z \not\leq b_{Q'}$ for some z. Then for $x \in F$, $z - x$ is orthogonal to the rows of Q'' but $q'_{j.}(z - x) > 0$ for some row of Q'. But then $x + \varepsilon(z - x) \in F$ for small enough $\varepsilon > 0$ and $q'_{j.} \cdot [x + \varepsilon(z - x)]$ is strictly increases if we increase ε contradicting the fact that it is constant δ_j.

□

A face consisting of a single point is called a **vertex**. An $x \in P$ is **extremal** if it is not a convex combination of other elements of P.

1.32 Theorem. *The following are equivalent.*

1. *x^* is a vertex,*

2. *x^* is extremal,*

3. *x^* is a basic solution and $T(P) = \{\underline{0}\}$.*

Proof. 1 $\implies$ 2: suppose that x^* is the only optimal solution with respect to c. Assume, to the contrary, that $\sum_{i=1}^k \lambda_i x_i = x^*$, where $x_i \in P \setminus \{x^*\}$, $\lambda_i > 0$ and $\sum_{i=1}^k \lambda_i = 1$. Then

$$cx^* = c\left(\sum_{i=1}^k \lambda_i x_i\right) = \sum_{i=1}^k \lambda_i c x_i < \sum_{i=1}^k \lambda_i c x^* = cx^*.$$

2 $\implies$ 3: let $P = P(Q, b)$. We know that $Q^=_{x^*} x = \underline{0}$ has no nontrivial solution, otherwise for such a solution z for a small enough $\varepsilon > 0$ we would have $x^* - \varepsilon z,\ x^* + \varepsilon z \in P$, thus $x^* = \frac{1}{2}(x^* - \varepsilon z) + \frac{1}{2}(x^* + \varepsilon z)$. Hence $r(Q^=_{x^*})$ must be the number of columns and so is $r(Q)$.

3 $\implies$ 1: $T(P) = \{\underline{0}\}$ implies that the columns of Q are linearly independent and so does the columns of $Q^=_{x^*}$ (the number of columns are their rank). Let c be the sum of the rows in $Q^=_{x^*}$. Then maximizing cx means satisfying $Q^=_{x^*} x = b_{Q^=_{x^*}}$ which cannot other solution than x^* because of the independence of the columns. □

1.33 Claim. *A nonempty polyhedron P has a vertex if and only if $T(P) = \{\underline{0}\}$.*

Proof. The necessity of $T(P) = \{\underline{0}\}$ follows from 3. of Theorem 1.32. Suppose $T(P) = \{\underline{0}\}$ and take a strong basic solution (exists by 1.19). It is a vertex by 3. of Theorem 1.32. □

1.34 Claim. *A nonempty polyhedron P is the convex hull of its vertices if and only if $D(P) = \{\underline{0}\}$.*

Proof. Assume that P is the convex hull of its vertices. By 3. of Theorem 1.32, the vertices are strong basic solutions, thus there are just finitely many (see Corollary 1.22). Therefore P must be bounded which implies $D(P) = \{\underline{0}\}$.

Suppose $D(P) = \{\underline{0}\}$. Then $T(P) = \{\underline{0}\}$, thus every basic solution is strong and hence is a vertex. Let $x \in P$ be arbitrary. By induction on $r(x)$, we show that x is a convex combination of vertices. If $r(x) = 0$, then x itself is a vertex. Let $r(x) > 0$ and assume $P = P(Q, b)$. Let q be a row of Q which is not spanned by the rows of $Q^=_x$. By Fredholm alternative theorem 1.7 there is some z for which $Q^=_x z = \underline{0}$ and $zq > 0$. Then $w := x + \lambda z \in P$ and $r(w) < r(x)$ for a suitable $\lambda > 0$. We know that $Q(-z) \not\leq \underline{0}$ since $D(P) = \{\underline{0}\}$ and $z \neq \underline{0}$. Hence there is a row p of Q such that $-pz > 0$. Then $v := x - \mu z \in P$ and $r(v) < r(x)$ for a suitable $\mu > 0$. By the induction hypothesis, w, v are the convex combination of the vertices of P. But then x as well, since it is on the segment$[w, v]$. □

2 TU matrices

A matrix is **totally unimodular** (shortly TU) if all of its subdeterminants are in $\{0, 1, -1\}$. Let I be the **identity matrix** i.e., the matrix that has 1 in the diagonal and 0 elsewhere. We define $\boldsymbol{e_i}$ to be the unit vector which has one 1 at the i-th coordinate and the other coordinates are 0 (the size of I and e_i will be clear from the context).

2.1 Observation. TU matrices are invariant under the following operations:

1. transpose,
2. deletion of rows (columns),
3. changing the order of rows (columns),
4. multiplying a row or column by -1,
5. adding a new row (column) which is identical with some original,
6. adding a new row (column) of form e_i (use Laplace expansion to show the result is TU)
7. adding a identity matrix bellow (next to) the matrix.

Examples

The **incidence matrix** of a digraph $D = (V, A)$ is the following. The columns are corresponding to the edges and the rows to the vertices. In the row corresponding to $uv \in A$ we have a 1 at v and a -1 at u and the other coordinates are 0. The incidence matrix of an undirected graph is similar except there is a 1 at coordinate u at well.

2.2 Claim. *The incidence matrix of a digraph is TU.*

Proof. We show that every subdeterminant is $1, -1$ or 0 by induction on the size of the subdeterminant. The 1×1 case is clear. Suppose we know that the subdeterminants of size at most k are in $\{\pm 1, 0\}$. Take a submatrix Q of size $(k+1)\times(k+1)$. If there is a column with at most one non-zero element, then we use Laplace expansion with this column and the induction hypothesis. Otherwise in every column we have a 1 and a -1 and the other elements are 0, thus the sum of the rows is $\underline{0}$ which means the determinant is 0. $\square$

2.3 Corollary. *The incidence matrix of a bipartite graph is TU.*

Proof. Let $G = (S, T, E)$ be a bipartite graph. Multiply the rows corresponding to S by -1. The result is the incidence matrix of the digraph that we obtain by directing the edges of G from S to T. It is TU by Claim 2.2 and thus by Observation 2.1 the incidence matrix of G as well. $\square$

Let $D = (V, A)$ be a digraph (loops and multiple edges are allowed) and suppose that $T \subseteq A$ is a spanning tree in the undirected sense. Let Q be a matrix which rows are indexed by the elements of T and the columns by the edges in $A \setminus T$. Consider the column corresponding to an $e \in A \setminus T$. Then e creates a unique cycle in T. If some $f \in T$ is not in the cycle, then $Q_{f,e} := 0$. If f is in the cycle, then we consider the orientation of the cycle given by e. If the orientation of f agrees with this orientation, then $Q_{f,e} := 1$ otherwise $Q_{f,e} := -1$. Matrices that are representable this way are called **network matrices**.

2.4 Theorem. *Network matrices are TU.*

Proof. First we check that a submatrix of a network matrix is a network matrix. Indeed, the deletion of a column corresponding to $e \in A \setminus T$ means a deletion of e from D. The deletion of a row corresponding to $f \in T$ means the contraction of f in D.

Therefore it is enough to show that the determinant of a network matrix is in $\{\pm 1, 0\}$. Suppose that it is false, and take a smallest counterexample Q. Note that it remains a counterexample if we reverse some edges. Let $uv \in T$ where v is a leaf. If there is no edge in $A \setminus T$ which is incident with v, then the determinant is 0 (constan 0 row) which is impossible. If there is exactly one such an edge then we use Laplace expansion and get a contradiction since Q is a smallest counterexample and all submatrices of Q are network matrices.

Thus there must be distinct edges $e, f \in A \setminus T$ which are incident with v. By redirecting edges, we can assume that e enters v and f leaves v. Adding the column of f to the column of e does not change the determinant. On the other hand, the result is a network matrix defined by T and the digraph that we obtain by the deletion of e and drawing a new edge e' from the tail of e to the head of f. In the new counterexample defined by T and D' the number of edges incident with v is strictly less. Iterating this we get a counterexample in which at most one edge is incident with v which we already know cannot exist. $\square$

Properties of TU matrices

We say that x is an **integral vector** if all of its coordinates are integers. An **integral polyhedron** is a nonempty polyhedron such that every face ot it contains an integral vector. In other words, whenever cx is bounded from above on it, then there is an integral optimal solution.

2.5 Theorem. *If $P(Q, b) \neq \varnothing$, where Q is TU and b is integral, then it is an integral polyhedron.*

Proof. Consider a face defined by the objective function c. There is an optimal solution x^* which is a strong basic solution (see Theorem 1.23). We have seen (Claim 1.21) that there is some quadratic, regular submatrix Q' of Q with $r(Q') = r(Q)$ such that one can get x^* by extending the unique solution of $Q'x = b_{Q'}$ with zeroes. Since $\det(Q') \in \{1, -1\}$ and $b_{Q'}$ is an integral vector, it follows from the Cramer's rule that the unique solution of $Q'x = b_{Q'}$ is an integral vector and therefore x^* as well. $\square$

2.6 Lemma. *Let Q be TU. If there is a $x_0 \in P(Q)$ with $cx_0 > 0$, then there is a $\pm 1, 0$ valued $x^* \in P(Q)$ with $cx^* > 0$.*

Proof. The system

$$\max cx \quad -1 \leq x_i \leq 1 \quad (i = 1, \ldots, n), \; Qx \leq \underline{0}$$

defined by a TU matrix (see Observation 2.1) and the bounding vector is integer, thus there is an integral optimal solution x^* by Theorem 2.5. For a small enough $\varepsilon > 0$ the vector εx_0 is a solution of the new system and $c(\varepsilon x_0) = \varepsilon(cx_0) > 0$, hence $cx^* > 0$. $\square$

2.7 Corollary (Farkas lemma TU-version)**.** *If $Qx \leq b$ is unsolvable where Q is TU (b is arbitrary), then there is a $1, 0$ valued y for which $yQ = \underline{0}$ and $yb < 0$.*

Proof. By applying the general form of Farkas lemma (i.e., exercise 1.8 with $D := Q, b_1 := b$) we obtain a $y_1 \geq \underline{0}$ with $y_1 Q = \underline{0}, y_1 b < 0$. Then use the previous lemma where the matrix is the transpose of Q and $-Q$ on top of each other, $x_0 := y_1$ and $c := -b$. $\square$

For $\alpha \in \mathbb{R}$ and $X \subseteq \mathbb{R}^n$ let $\boldsymbol{\alpha X} = \{\alpha x : x \in X\}$. Note that for an integer $k > 0$ we have $kP(Q,b) = P(Q,kb)$. The polyhedron P has the **integer decomposition property** if for every integer $k > 0$ and every integral $z \in kP$, there are integral vectors $z_1, \ldots, z_k \in P$ such that $z = \sum_{i=1}^k z_i$.

2.8 Theorem. *If $P = P(Q,b) \neq \varnothing$, where Q is TU and b is integral, then P has the integer decomposition property.*

Proof. Suppose that $z \in kP$ is integral. Remember that $kP = P(Q,kb)$. We apply induction on k. For $k = 1$, the choice $z_1 := z$ is appropriate. Suppose $k > 1$. If we can find an integral z_1 for which $Qz_1 \leq b$ and $Q(z - z_1) \leq (k-1)b$, then we are done since by induction we have a desired decomposition $z_2, \ldots, z_k$ of $z - z_1 \in (k-1)P$. The conditions for z_1 form a linear inequality system which matrix is still TU (Observation 2.1), the bounding vectors are integral and $\frac{1}{k}z$ is a solution. Therefore by Theorem 2.5, we have an integral solution as well which is suitable for z_1. □

2.9 Corollary. *Let Q be TU and $b_1, b_2 \in \mathbb{Z}^n$. If $kb_1 \leq Qz \leq kb_2$ where $z \geq \underline{0}$ is integral, then there are z_i $(i = 1, \ldots, k)$ integral vectors such that $z = \sum_{i=1}^k z_i$ and $z_i \geq \underline{0}$, $b_1 \leq Qz_i \leq b_2$.*

The integer n is a **rounding** of $\alpha \in \mathbb{R}$ if $n \in \{\lfloor \alpha \rfloor, \lceil \alpha \rceil\}$. The integral vector z is a rounding of x if z_i is a rounding of x_i for every i. For a vector x, we define $\lfloor x \rfloor$ and $\lceil x \rceil$ coordinate-wise.

2.10 Theorem. *Let Q be TU and $y \in \mathbb{R}^n$. Then there is a rounding z of y for which Qz is a rounding of Qy.*

Proof. The matrix of the system

$$\lfloor y \rfloor \leq x \leq \lceil y \rceil$$
$$\lfloor Qy \rfloor \leq Qx \leq \lceil Qy \rceil$$

is TU and y is a solution, thus there is an integral solution z by Theorem 2.5. □

2.11 Theorem. *If Q is TU, then for every $k \geq 2$, there is a k-colouring of the columns which is uniform in the following sense: in every row the sums of the entries corresponding to the same colour-class differ by at most one.*

Proof. Let $d := Q\underline{1}$ (i.e., the sum of the columns of Q) and consider the system

$$k\left\lfloor \frac{d}{k} \right\rfloor \leq Qx \leq k\left\lceil \frac{d}{k} \right\rceil, \; x \geq \underline{0}.$$

By the construction is clear that $x = \underline{1}$ is a solution, thus by Corollary 2.9 we have z_i $(i = 1, \ldots, k)$ with

$$z_i \geq \underline{0},$$
$$\sum_{i=1}^k z_i = \underline{1},$$
$$\left\lfloor \frac{d}{k} \right\rfloor \leq Qz_i \leq \left\lceil \frac{d}{k} \right\rceil.$$

It follows that z_i are 1,0 valued and choosing z_i to be the characteristic function of the i-th colour-class is appropriate. □

Applications of TU matrices

Applications in bipartite graphs

2.12 Theorem (Kőnig)**.** *In a bipartite graph $G = (V,E)$ the maximal number of independent edges is equal to the minimal number of vertices covering all the edges.*

Proof. The incidence matrix Q of G is TU by Corollary 2.3. Let $x \in \mathbb{R}^E$ be a variable and consider the linear program

$$\max \sum_{e \in E} x(e) \quad x \geq \underline{0}, \; Qx \leq \underline{1}.$$

The matrix of the system is TU (see Observation 2.1) thus by Theorem 2.5 there is an optimal solution x^* which is integral. It means in this case that x^* is $1,0$ valued. Clearly $\{e \in E : x^*(e) = 1\}$ is a matching in G (because $Qx^* \leq \underline{1}$) i.e., x^* is a characteristic vector of a matching M. Note that $\sum_{e \in E} x^*(e) = |M|$. The dual problem is the following (use exercise1.27):

$$\min \sum_{v\in V} y(v) \quad y \geq \underline{0},\ yQ \geq \underline{1}.$$

We can take an optimal solution y^* which is integral for the same reason as in the case of x^*. Note that y^* cannot have a coordinate which is larger than 1 since reducing it to one we would get a better solution of the dual. Thus y^* is 1,0 valued and it is the characteristic vector of a $U \subseteq V$ that covers the edges (because $y^*Q \geq \underline{1}$) and by the Strong Duality Theorem 1.25, $|U| = \sum_{v\in V} y^*(v) = \sum_{e\in E} x^*(e) = |M|$. □

2.13 Theorem (Egerváry)**.** *Let $G = (V, E)$ be a bipartite graph and $w \in \mathbb{R}^E$. A weighted cover is a $\pi \in \mathbb{R}^V$ such that $\pi \geq 0$ and $\pi(u) + \pi(v) \geq w(uv)$ for every $uv \in E$. Then the maximal possible total weight of a matching (weight of a matching M is $\sum_{e\in M} w(e)$) equals to the minimal possible total weight of a weighted cover (total weight of π is $\sum_{v\in V} \pi(v)$).*

Proof. Consider the linear program

$$\begin{aligned} &\max wx \\ &x \geq \underline{0} \\ &Qx \leq \underline{1}. \end{aligned}$$

Take an optimal solution x^* which is integral (it is possible by Theorem 2.5). It is a characteristic function of a matching M with total weight wx^*. Because of Strong Duality 1.25, wx^* is equal to

$$\begin{aligned} &\min \pi\underline{1} \\ &\pi \geq \underline{0} \\ &\pi Q \geq w. \end{aligned}$$

Observe that the conditions say that π is a weighted cover and $\pi\underline{1}$ is its total weight. □

2.14 Theorem. *Let $G = (V, E)$ be a bipartite graph and let k be a positive integer. Then there is a partition of $F_1, \dots, F_k$ of E such that $|F_i|$ is a rounding of $\frac{|E|}{k}$, and $d_{F_i}(v)$ is a rounding of $\frac{d_E(v)}{k}$ for every $v \in V$.*

Proof. Extend the incidence matrix of G by a constant 1 row. It is enough to show that the resulting matrix Q is TU because then we are done by applying Theorem 2.11. We show that Q is a network matrix (which is enough by Theorem 2.4). Let U, W be a bipartition of V such that every edge goes between U and W. Direct the edges from U to W. Exactly These edges will be not in the spanning tree at the end of the construction. Pick two new vertices u and w. From every $v \in W$ draw a directed edge to w and from u draw a directed edge to every $v \in U$, finally add the edge wu. The construction is done. If $ab \in E$ with $a \in U$ and $b \in W$, then the corresponding unique cycle is a, b, w, u thus in this column there is a 1 at the rows: ua, bw, wu. □

2.15 Corollary. *If $G = (V, E)$ is a bipartite graph with maximal degree k, then one can partition E into k many matchings where the size of each is a rounding of $\frac{|E|}{k}$.*

The number of edges spanned by a set $Z \subseteq V$ is denoted by $\boldsymbol{i_G(Z)}$. We omit G if it is clear from the context.

2.16 Theorem. *Let $G = (V, E)$ be a bipartite graph. There are k many pairwise disjoint matchings of size l if and only if for every $Z \subseteq V$ we have $i(Z) \geq k(l - |V \setminus Z|)$.*

Proof. If there are k many pairwise disjoint matchings of size l, then any of them has at most $|V \setminus Z|$ edges which are incident with $V \setminus Z$ and therefore at least $l - |V \setminus Z|$ which are not (i.e., spanned by Z) thus $i(Z) \geq k(l - |V \setminus Z|)$ must hold.

Suppose that for every $Z \subseteq V$ we have $i(Z) \geq k(l - |V \setminus Z|)$. Let Q be the incidence matrix of G. The matrix of the system

$$\begin{aligned} &\max \underline{1}x \\ &\underline{0} \leq x \leq \underline{1} \\ &Qx \leq \underline{k} \end{aligned}$$

is TU (see Observations 2.3 and 2.1). By Theorem 2.5, it is an integral polyhedron, thus we can take an integral optimal solution x^*. If $\underline{1}x^* \geq kl$, then $F := \{e \in E : x^*(e) = 1\}$ is a subgraph with maximal degree at most k and with at least kl edges. Hence by Corollary 2.15, we can partition F into k many matchings where all of them has size at least l.

Suppose, to the contrary, that the the optimum is less than kl. Consider the dual.

$$\begin{aligned}&\min\ y\underline{k} + z\underline{1},\\&(y,z) \geq \underline{0},\\&yQ + z \geq \underline{1}.\end{aligned}$$

The dual polyhedron is also integral thus we can choose an integral optimal dual solution (y^*, z^*). Because of the optimality it cannot have values larger than 1 thus it is 1,0 valued. Let $Z^* := \{v \in V\ :\ y^*(v) = 0\}$. Note that by optimality $z^*(e) = 1$ if and only if e is spanned by Z^* (hence $\underline{1}z = i(Z^*)$). Applying Strong Duality Theorem 1.25, the dual optimum is smaller than kl i.e.,

$$k\,|Z^* \setminus V| + i(Z^*) < kl.$$

It contradicts the assumption that $i(Z) \geq k(l - |V \setminus Z|)$ holds for $Z \subseteq V$ □

Applications in digraphs

For a digraph $D = (V, A)$ and $Z \subseteq V$ let us denote the set of the ingoing and outgoing edges by $\mathbf{in}_{\boldsymbol{A}}\boldsymbol{(Z)}$ and $\mathbf{out}_{\boldsymbol{A}}\boldsymbol{(Z)}$ respectively. Let $\boldsymbol{\varrho_A(Z)} := |\mathsf{in}_A(Z)|$ and $\boldsymbol{\delta_A(Z)} := |\mathsf{out}_A(Z)|$. Finally if $x \in \mathbb{R}^A$, then $\boldsymbol{\varrho_x(Z)}$ and $\boldsymbol{\delta_x(Z)}$ are defined to be $\sum_{e\in \mathsf{in}_A(Z)} x(e)$ and $\sum_{e\in \mathsf{out}_A(Z)} x(e)$ respectively.

2.17 Lemma. *If $D = (V, A)$ is a digraph, $x \in \mathbb{R}^A$ and $Z \subseteq V$, then $\sum_{v\in Z}(\varrho_x(v) - \delta_x(v)) = \varrho_x(Z) - \delta_x(Z)$.*

Proof. If $uv \in A$ with $u, v \in Z$, then $x(uv)$ appears with $+$ sign at $\varrho_x(v)$ and with $-$ at $\delta_x(u)$. If $u, v \notin Z$, then $x(uv)$ does not appear at the sum. If $v \in Z$ but $u \notin Z$, then it appears only once at $\varrho_x(v)$ with $+$. If $u \in Z$ but $v \notin Z$, then it appears only once at $\delta_x(u)$ with $-$. □

An $x \in \mathbb{R}^A$ is called a **circulation** if $\varrho_x(v) = \delta_x(v)$ for $v \in V$.

2.18 Lemma. *$\forall v \in V : \varrho_x(v) = \delta_x(v)$ if and only if $\forall v \in V : \varrho_x(v) \leq \delta_x(v)$.*

Proof. Assume that $\forall v \in V : \varrho_x(v) \leq \delta_x(v)$. Then (applying Lemma 2.17)

$$0 \leq \sum_{v\in V}(\delta_x(v) - \varrho_x(v)) = \delta_x(V) - \varrho_x(V) = 0 - 0 = 0,$$

thus $\delta_x(v) - \varrho_x(v) = 0$ for $v \in V$. □

2.19 Theorem (Hoffman)**.** *Let $D = (V, A)$ be a digraph and $f, g \in \mathbb{R}^A$ with $f \leq g$. There is a circulation x in D with $f \leq x \leq g$ if and only if $\varrho_f(Z) \leq \delta_g(Z)$ holds for all $Z \subseteq V$. If f, g are integral and there is a feasible circulation, then the set of the feasible circulations form an integral polyhedron.*

Proof. If x is a circulation with $f \leq x \leq g$ and $Z \subseteq V$, then

$$0 = \sum_{v\in Z}(\varrho_x(v) - \delta_x(v)) = \varrho_x(Z) - \delta_x(Z) \geq \varrho_f(Z) - \delta_g(Z).$$

Let Q be the incidence matrix of D. By applying Apply Lemma 2.18, we conclude that the feasible circulations are exactly the elements of the following polyhedron (which is integral by Theorem 2.5 unless it is empty).

$$\begin{aligned}&Qx \leq \underline{0}\\&f \leq x \leq g\end{aligned}$$

If the polyhedron above is empty, then by Corollary 2.7 the following system has a $1, 0$ valued solution.

$$yQ + z - s = \underline{0} \qquad (1)$$
$$zg - sf < 0 \qquad (2)$$

Let (y^*, z^*, s^*) be such a solution. We may assume that there is no $e \in A$ with $z^*(e) = s^*(e) = 1$ otherwise by changing both $z^*(e)$ and $s^*(e)$ to 0 gives an other $1, 0$ valued solution (where we use $f \leq g$ to prevent the violation of (3)). Let $Z := \{v \in V\ :\ y^*(v) = 1\}$. From the previous assumption and (1) we may conclude that z^* is the characteristic vector of $\mathsf{out}_A(Z)$ and s^* is the characteristic vector of $\mathsf{in}_A(Z)$. But then by (3) we obtain $\varrho_f(Z) - \delta_g(Z) > 0$. □

Let $D = (V, A)$ be a digraph with $s \neq t \in V$. Suppose that s has no ingoing and t has no outgoing edges. An $\boldsymbol{s \to t}$ **flow** is an $x : A \to \mathbb{R}_+$ such that $\varrho_x(v) = \delta_x(v)$ for $v \in V \setminus \{s, t\}$. The **amount** of the flow is $\delta_x(s)$. The flow x is called **feasible** with respect to a $g : A \to \mathbb{R}^+$ if $x \leq g$. An $\boldsymbol{s \to t}$ **cut** is a $C \subseteq A$ that covers all the $s \to t$ paths in D. The value $g(C)$ of such a C is defined to be $\sum_{e \in C} g(e)$.

2.20 Theorem (Ford-Fulkerson)**.** *Let $D = (V, A)$ be a digraph with $s \neq t \in V$. Suppose that s has no ingoing and t has no outgoing edges and let $g : A \to \mathbb{R}_+$. The maximal amount feasible $s \to t$ flows is equal to the minimal value of $s \to t$ cuts i.e.,*

$$\max\{\delta_x(s) \,:\, x \text{ is a feasible } s \to t \text{ flow with respect to } D, g\} = \min\{g(C) \,:\, C \text{ is an } s \to t \text{ cut in } D\}.$$

Furthermore, if g is integral, then x can be chosen integral.

Proof. Let us denote the minimal value of $s \to t$ cuts by $\boldsymbol{\lambda_g(s, t)}$. Extend D with the new edge ts to obtain $D' = (V, A')$. We define f to be 0 on A. Let $f(ts) := g(ts) := \lambda_g(s, t)$. Observe that if x is a feasible circulation with respect to D', f, g, then the restriction of x to A is an $s \to t$ flow of amount $\lambda_g(s, t)$. We show that the desired feasible circulation exists by applying Theorem 2.19. Let $Z \subseteq V$ be arbitrary. If ts does not enter Z, then $\varrho_f(Z) = 0 \leq \delta_g(Z)$ since $g : A' \to \mathbb{R}^+$. If ts enters Z, then

$$\varrho_f(Z) = f(ts) = \lambda_g(s, t) = \min\{\delta_g(X) \,:\, X \subseteq V, s \in X,\ t \notin X\} \leq \delta_g(Z).$$

□

2.21 Corollary (Menger; edge-version, directed)**.** *The maximal number of edge-disjoint $s \to t$ paths in the digraph D is equal to the minimal size of the $s \to t$ cuts in D.*

Proof. Take an integral $s \to t$ flow x of maximal amount which is feasible with respect to $g \equiv 1$. One can partition the set $\{e \in A \,:\, x(e) = 1\}$ by the greedy method into $\lambda_D(s, t)$ many directed $s \to t$ paths and directed cycles. □

If $D = (V, A)$ is a digraph, then a cost function $c \in \mathbb{R}^A$ is called **conservative** if there is no directed cycle with negative total cost. Functions $\pi : V \to \mathbb{R}$ are called **potentials**. The **tension** $\boldsymbol{\Delta_\pi} \in \mathbb{R}^A$ induced by π maps a $u \to v$ edge e to $\pi(v) - \pi(u)$. A potential π is **feasible** with respect to c if $\Delta_\pi \leq c$.

2.22 Excercise. In any digraph $D = (V, A)$ the tensions form a linear subspace of $\mathbb{R}^A$ and the circulations are the orthogonal complement of this subspace. (Hint: let Q be the incidence matrix of D, then tensions are generated by the rows and circulations is the kernel.)

2.23 Theorem (Gallai)**.** *Let $D = (V, A)$ be a digraph and $c \in \mathbb{R}^A$. There is a feasible potential π if and only if c is conservative. If c is conservative and integral, than π can be chosen integral.*

Proof. Let Q be the incidence matrix of D. By Corollary 2.7, the system $\pi Q \leq c$ has no solution if and only if $\exists x \in \{0, 1\}^A$ with $Qx = \underline{0}$ and $cx < 0$. Let $C := \{e \in A \,:\, x(e) = 1\}$. Then $Qx = \underline{0}$ means $\varrho_C(v) = \delta_C(v)$ for $v \in V$. By the greedy method one can partition C into directed cycles. At least one such a cycle has negative total cost since $cx < 0$. The last sentence of the theorem follows from Theorem 2.5. □

2.24 Theorem (Duffin)**.** *Let $D = (V, A)$ be a digraph such that $t \in V$ is reachable from $s \in V$. Let $c \in \mathbb{R}^A$ be conservative. Then*

$$\max\{\pi(t) - \pi(s) \,:\, \pi \text{ is a feasible potential}\} = \min\{c(P) \,:\, P \text{ is a directed } s \to t \text{ path}\}.$$

Proof. Let Q be the incidence matrix of D. The maximum on the left can be expressed as

$$\begin{aligned} &\max \pi(t) - \pi(s) \\ &\pi Q \leq c. \end{aligned}$$

By the Duality Theorem 1.25 it is equal to

$$\min cx \tag{1}$$

$$x \geq \underline{0} \tag{2}$$

$$(Qx)(v) = \begin{cases} -1 & \text{if } v = s \\ 1 & \text{if } v = t \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

By Theorem 2.5, we have an integral optimal solution x^* of the dual. It defines a multi-subset of the edges (because of (2)) which we can be partitioned by (3) into a directed $s \to t$ path P and some directed cycles. These cycles must have 0 total cost since c is conservative and x^* is optimal, thus $cx^* = c(P)$. □

2.25 Theorem. *Let $D = (V, A)$ be a strongly connected digraph and $f, g, c \in \mathbb{R}^A$ with $f \leq g$. Suppose that there exists a feasible circulation x. Let us define the following digraph $D_x = (V, A_x)$ and cost function $c_x \in \mathbb{R}^{A_x}$. If $x(e) < g(e)$, then $e \in A_x$ with $c_x(e) := c(e)$. If $f(e) < x(e)$, then put the reverse $\overleftarrow{e}$ of e in A_x with $c_x(\overleftarrow{e}) := -c(e)$ (parallel edges may occur). The following are equivalent.*

1. *x minimize cx among the feasible circulations.*
2. *c_x is conservative.*
3. *There exists a $\pi \in \mathbb{R}^V$ (which can be chosen integral if c is integral) such that $\pi(v) - \pi(u) \leq c(e)$ whenever $e \in A$ is a $u \to v$ edge with $x(e) < g(e)$ and $\pi(v) - \pi(u) \geq c(e)$ if $e \in A$ is a $u \to v$ edge with $x(e) > f(e)$.*

Proof. $1 \implies 2$: If c_x is not conservative, then take a cycle C in D_x with $c_x(C) < 0$ and take a $\varepsilon > 0$. For $e \in A \cap C$ increase $x(e)$ by ε. For $\overleftarrow{e} \in C$ decrease $x(e)$ by ε. The resulting x' is a circulation and it is feasible if ε is small enough. Furthermore, $cx' = cx + \varepsilon c_x(C) < cx$ thus 1 is false.

$2 \implies 3$: By Theorem 2.23, there is a feasible potential π with respect to c_x. From the construction of D_x it follows that π satisfies the desired properties.

$3 \implies 1$: Let x' be a feasible circulation with respect to D, f, g and let π be as in 3. Let $\boldsymbol{c_\pi} := c - \Delta_\pi$. By Exercise 2.22, $\Delta_\pi x' = 0$ and therefore $cz = c_\pi z$ for every circulation z. Note that 3 says $c_\pi(e) \geq 0$ if $x(e) < g(e)$ and $c_\pi(e) \leq 0$ if $x(e) > f(e)$ for every $e \in A$.

$$
\begin{aligned}
&cx' = c_\pi x' = \sum_{c_\pi(e)>0} c_\pi(e)x'(e) + \sum_{c_\pi(e)<0} c_\pi(e)x'(e) \geq \\
&\sum_{c_\pi(e)>0} c_\pi(e)f(uv) + \sum_{c_\pi(e)<0} c_\pi(e)g(e) = \\
&\sum_{c_\pi(e)>0} c_\pi(e)x(e) + \sum_{c_\pi(e)<0} c_\pi(e)x(e) = c_\pi x = cx
\end{aligned}
$$

□

Let $D = (V, A)$ be a strongly connected digraph. We are looking for a shortest possible tour which goes through all the edges at least once and arrives at the starting vertex.

2.26 Theorem (Chinese Postman)**.** *Let $D = (V, A)$ be a strongly connected digraph. The minimum of $|F|$, where F consists of edges each of them parallel to an edge in A and $D = (V, A \cup F)$ has an Euler tour, is equal to*

$$\max\{\sum_{i=1}^{k} \delta_A(V_i) - \varrho_A(V_i) \;:\; k > 0,\; V \supseteq V_1 \supseteq \ldots, \supseteq V_k,\; \textit{no edge in } A \textit{ enters more than one } V_i\}.$$

Proof. Observe that the maximum does not change if we demand $\delta_A(V_i) - \varrho_A(V_i) \geq 0$ for every i. To show the min $\geq$ max direction, observe that for a desired F we have $\varrho_{A\cup F}(Z) = \delta_{A\cup F}(Z)$ for every $Z \subseteq V$. Thus every V_i needs to get at least $\delta_A(V_i) - \varrho_A(V_i)$ new ingoing edges. Since no edges enters more than one V_i, these edge sets are pairwise disjoints.

To show max=min, let f and c be the constant 1 function on A and let g be constant $|A|^2 + 1$ on A (any large enough number is good). Note that any integral feasible circulation gives a desired $A \cup F$. If we pick a directed cycle C_e through e for $e \in A$, then the sum z of the characteristic vectors of these cycles is a feasible circulation with $cz \leq |A|^2$. By Theorem 2.19, there is an integral feasible circulation x^* that minimize cx. We know that $x^*(e) < |A|^2 + 1 = g(e)$ holds for $e \in A$ otherwise z would be a better solution. By Theorem 2, there is a $\pi \in \mathbb{R}^V$ such that $\pi(v) - \pi(u) \leq 1$ for any $u \to v$ edge $e \in A$ and $\pi(v) - \pi(u) \geq 1$ if e is an $u \to v$ edge with $x(e) > f(e)$. By adding a constant to π, we can assume that the smallest value of π is 0. Let us denote the largest value of π by k and let $V_i := \{v \in V \;:\; \pi(v) \geq i\}$ for $i = 1, \ldots, k$. We have $\pi(v) \leq \pi(u) + 1$ whenever there is an edge from u to v in A, i.e., each edge enters at most one V_i. Furthermore, if a $u \to v$ edge e has been multiplied, then $\pi(v) = \pi(u) + 1$, i.e., e enters some V_i. This means that the number of new edges given by x^* is exactly $\sum_{i=1}^{k} \delta_D(V_i) - \varrho_D(V_i)$. □

A further application

2.27 Theorem. *For any sequence (a_n) in $\mathbb{R}$ there is a sequence (b_n) in $\mathbb{Z}$ such that for every $k \leq m$ the sum $\sum_{i=k}^{m} b_i$ is a rounding of $\sum_{i=k}^{m} a_i$.*

Proof. It is enough to prove the statement for a finite sequence $a_1, \dots, a_N$ since the infinite version follows by König's lemma. Consider the digraph D on $v_1, \dots, v_N$ where $v_i v_j$ is an edge if either $j = i + 1$ or $j < i$. Let T consists of the edges $v_i v_{i+1}$ and let Q be the network matrix defined by D and T. Let $a := (a_1, \dots, a_N)$, then the theorem says that there is a rounding b of a for which bQ is a rounding of aQ. Since Q is TU by Theorem 2.4, it follows from Theorem 2.10.

□

3 Total dual integrality

3.1 Theorem. *Let $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. There is no $x \in \mathbb{Z}^n$ for which $Ax = b$ if and only if there is a $y \in \mathbb{R}^m$ for which yA is integral but $yb \notin \mathbb{Z}$.*

Proof. Suppose that there is a $y \in \mathbb{R}^m$ for which yA is integral and $yb \notin \mathbb{Z}$. Assume for a contradiction that $Ax = b$ for some $x \in \mathbb{Z}^n$. Multiply both sides of '$Ax = b$' by y. Then the left side is an integer since yA and x are both integral but the right side is not which is a contradiction.

Assume now that there is no $y \in \mathbb{R}^m$ for which yA is integral and $yb \notin \mathbb{Z}$. Consider first the case when A is quadratic and regular. We need to show that $x := A^{-1}b$ is integral. Suppose that $x_i \notin \mathbb{Z}$. Let y be the i-th row of A^{-1}. Then yA is integral but yb is not which contradicts our assumption. There is an $x \in \mathbb{R}^n$ for which $Ax = b$, otherwise by applying Fredholm alternative theorem 1.7, we know that the system $yA = \underline{0}$, $yb = \frac{1}{2}$ is solvable. We can assume that the rows of A are linearly independent (and hence $m \leq n$), otherwise we keep just a base of the row space knowing that any solution of the corresponding subsystem is a solution of the whole system and there is still no y for the smaller system (otherwise by extending it with 0 coordinates there would be for the original as well).

Consider the following column operations: changing the order of the columns, negating columns, adding n-times ($n \in \mathbb{Z}$) a column to another column. Note that if we obtain A' from A by using these operations, then there is no $y \in \mathbb{R}^m$ for which yA' is integral and $yb \notin \mathbb{Z}$ since the same y would be good for A as well. On the other hand, if $A'x = b$ has an integral solution, then we can get an integral solution of $Ax = b$. Note that $r(A) = r(A')$ thus the rows of A' are also independent.

We can transform A to a form (applying the column operations) such that in the first row exactly the first entry is non-zero. Observe that in the second row we must have a non-zero entry in the last $n - 1$ columns otherwise the first two rows would be linearly dependent. By using the column operations with the last $n-1$ columns, we can reach a matrix A' such that $A'_{2,2} \neq 0$ and $A'_{2,i} = 0$ for $i > 2$. Following this method, we obtain an A'' with $A''_{i,i} \neq 0$ and $A''_{i,j} = 0$ for $j > i$. It means that the submatrix B consisting of the first m columns of A'' form a lower triangular matrix with non-zero elements in the diagonal and therefore regular. There is no $y \in \mathbb{R}^m$ for which yB is integral and $yb \notin \mathbb{Z}$ because the same y would be good for A'' as well. We know from the regular case that there is an $x \in \mathbb{Z}^m$ for which $Bx = b$. By extending this x with arbitrary integers to an $x^* \in \mathbb{Z}^n$, we have $A''x^* = b$. By the properties of the column operations, it means that there is some $z \in \mathbb{Z}^n$ with $Az = b$. □

3.2 Theorem (Edmonds and Giles)**.** *Let $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ where $P := P(A, b) \neq \varnothing$. Then P is an integral polyhedron if and only if for every $c \in \mathbb{Z}^n :\ \sup\{cx :\ x \in P\} \in \mathbb{Z} \cup \{+\infty\}$.*

Proof. If P is integral and cx is bounded from above, then there is an integral optimal solution x^* (the face defined by c has an integral point). Thus if c is integral, then $cx^* \in \mathbb{Z}$.

Assume that the $\subseteq$-minimal face F of P does not contain integral points. Applying Claim 1.31, $F = \{x \in \mathbb{R}^n : A'x = b_{A'}\}$ where A' consists of some of the rows of A. By Theorem 3.1, there is a y such that $yA' =: c$ is integral but $yb_{A'} \notin \mathbb{Z}$. We can assume that $y \geq \underline{0}$ since $y + n\underline{1}$ also has the desired properties for every $n \in \mathbb{N}$. Then an x^* is an optimal solution with respect to c and P iff $x^* \in F$, furthermore, the value cx^* of the optimum is $cx^* = (yA')x^* = y(A'x^*) = yb_{A'} \notin \mathbb{Z}$ which contradicts the assumption. □

The linear inequality system $Ax \leq b$ ($A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$) is **totally dual integral** shortly **TDI** if it is solvable and for every $c \in \mathbb{Z}^n$ for which $\{cx : Ax \leq b\}$ is bounded from above the dual $\min\{yb : yA = c,\ y \geq \underline{0}\}$ has an integral optimal solution y^*.

3.3 Theorem. *If the system $Ax \leq b$ is TDI and $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$, then $P(A, b)$ is an integral polyhedron.*

Proof. Assume that $c \in \mathbb{Z}^n$ and cx is bounded on $P(A, b)$. By the Strong Duality theorem 1.25,

$$\max\{cx : Qx \leq b\} = \min\{yb : yQ = c,\ y \geq \underline{0}\}.$$

By the TDI property, there is an optimal integral dual solution y^*. Since $b \in \mathbb{Z}^m$, it implies that the value y^*b of the optimum is an integer. Then Theorem 3.2 ensures that $P(A, b)$ is an integral polyhedron. □

A function $m : 2^V \to \mathbb{R}$ is **modular** if $m(X)+m(Y) = m(X\cup Y)+m(X\cap Y)$ holds for every $X, Y \subseteq V$. Note that if m is modular and $m(\varnothing) = 0$, then $m(X) = \sum_{v\in X} m(v)$ for every $X \subseteq V$. A function $b : 2^V \to \mathbb{R}\cup\{+\infty\}$ is called **submodular** if for every $X, Y \subseteq V$ we have $b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y)$. If we demand this inequality only for X, Y with $X \cap Y \neq \varnothing$, then b is **intersecting submodular**. Finally b is **crossing submodular** if the inequality holds for **crossing** X and Y ($X, Y \subseteq V$ are crossing if $X \setminus Y,\ Y \setminus X,\ X \cap Y,\ V \setminus (X\cup Y)$ are all nonempty). A function $p : 2^V \to \mathbb{R}\cup\{-\infty\}$ is **supermodular** if $-p$ is submodular. Intersecting and crossing supermodular functions are defined analogously.

Submodular flows

Let $D = (V, A)$ be a digraph, $f \leq g \in \mathbb{Z}^A$ and let $b : 2^V \to \mathbb{Z} \cup \{+\infty\}$ be crossing submodular. For an $x \in \mathbb{R}^A$ and $Z \subseteq V$, let $\boldsymbol{\lambda_x(Z)} := \varrho_x(Z) - \delta_x(Z)$. Note that the function $\lambda_x : 2^V \to \mathbb{R}$ is modular and $\lambda_x(\varnothing) = \lambda_x(V) = 0$. We call an $x \in \mathbb{R}^A$ a **submodular flow** (shortly: subflow) if $\lambda_x(Z) \leq b(Z)$ for every $Z \subseteq V$. A subflow is feasible if $f \leq x \leq g$. For the further investigation of sublows we need some tools.

For a directed tree $T = (U, F)$ and $uv \in F$ let us denote by U_e the vertex set of the component of $T - uv$ that contains v. A **tree representation** of the set family $\mathcal{F} \subseteq 2^V$ consists of a directed tree $T = (U, F)$ and a function $\varphi : V \to U$ for which the inverse images of the sets U_{uv} with respect to φ are exactly the elements of $\mathcal{F}$ i.e., $\mathcal{F} = \{\varphi^{-1}(U_{uv}) : uv \in F\}$. A family $\mathcal{F} \subseteq 2^V$ is called **laminar** if for any $X, Y \in \mathcal{F}$ at least one of the following relations holds: $X \subseteq Y,\ X \supseteq Y,\ X \cap Y = \varnothing$. The set family is cross-free (with respect to the ground set V) if it does not contain crossing sets.

3.4 Excercise. Every laminar $\mathcal{F} \subseteq 2^V$ has a tree representation in which the directed tree is an arborescence.

3.5 Lemma. *Every cross-free $\mathcal{F} \subseteq 2^V$ has a tree representation.*

Proof. We can assume that $V \neq \varnothing$. Pick an $x \in V$ and replace every $Z \in \mathcal{F}$ that contains x with $V \setminus Z$. The resulting system $\mathcal{F}'$ is laminar thus by the previous exercise it has a tree representation. By reversing the tree-edges corresponding the elements of $\mathcal{F}' \setminus \mathcal{F}$ we obtain a tree representation of $\mathcal{F}$. □

3.6 Corollary. *Let $D = (V, A)$ be a digraph and let $\mathcal{F} \subseteq 2^V$ be cross-free. Consider the matrix $Q \in \{0, \pm1\}^{\mathcal{F}\times A}$ where $Q_{Z,e} = 1$ if $e \in \mathsf{in}_D(Z)$, $Q_{Z,e} = -1$ if $e \in \mathsf{out}_D(Z)$ and $Q_{Z,e} = 0$ otherwise. The matrix Q is TU.*

Proof. By Lemma 3.5, take a tree representation $T = (U, F)$, φ of $\mathcal{F}$. For each $uv \in A$, add the directed edge $\varphi(v)\varphi(u)$ to T. Consider the network matrix of the resulting system where the tree is F. On the one hand this matrix is exactly Q. On the other hand it is TU by Theorem 2.4. □

3.7 Theorem. *Let $D = (V, A)$ be a digraph and let $b : 2^V \to \mathbb{Z} \cup \{+\infty\}$ be crossing submodular. Suppose that $f, g : A \to \mathbb{Z}$ with $f \leq g$. Let $\mathcal{F}$ consists of those $Z \in 2^V$ for which $b(Z) < +\infty$. Consider the matrix $Q \in \{0, \pm1\}^{\mathcal{F}\times A}$ where $Q_{Z,e} = 1$ if $e \in \mathsf{in}_D(Z)$, $Q_{Z,e} = -1$ if $e \in \mathsf{out}_D(Z)$ and $Q_{Z,e} = 0$ otherwise.Then the following system is TDI if it is solvable.*

$$\begin{aligned} Qx &\leq b \\ f \leq x &\leq g. \end{aligned}$$

Proof. We need to find an integral optimal solution of the dual system for a given $c \in \mathbb{Z}^A$. For every $Z \in \mathcal{F}$, we have a dual variable $y(Z)$ and for each edge $e \in A$ we have two dual variables $y_f(e)$ and $y_g(e)$. The dual is (writing it in the long form) the following.

$$\min \sum_{Z\in\mathcal{F}} y(Z)b(Z) + \sum_{e\in A} y_g(e)g(e) - \sum_{e\in A} y_f(e)f(e) \tag{4}$$

$$\forall e \in A: \sum_{e\in \mathsf{in}_D(Z)} y(Z) - \sum_{e\in \mathsf{out}_D(Z)} y(Z) + y_g(e) - y_f(e) = c(e) \tag{5}$$

$$\forall Z \in \mathcal{F}:\ y(Z) \geq 0,\ \forall e \in A:\ y_g(e), y_f(e) \geq 0 \tag{6}$$

The strong basic solutions of the dual are rational-valued (see Claim 1.21), thus we have a rational-valued optimal solution (y', y'_g, y'_f) by Theorem 1.23. Let Q' be the submatrix of Q consisting of those rows of Q where y' is positive and let b' be the corresponding restriction of b. Note that every optimal solution of (we write it this time in the matrix form)

$$\min yb' + y_g \cdot g - y_f \cdot f \tag{7}$$

$$yQ' + y_g - y_f = c \tag{8}$$

$$(y, y_g, y_f) \geq \underline{0} \tag{9}$$

gives an optimal solution of (4)-(6) by extending y with 0 coordinates at the rows which are in Q but not in Q'.

Assume first that the **support** $\mathcal{F}_{y'}$ of y' (the set of those elements of $\mathcal{F}$ where y' is positive) is cross-free. Then by Corollary 3.6, Q' is TU and hence the matrix of the system (7)-(9) as well. Theorem 2.5 guarantees the existence of an optimal integral solution of (7)-(9). By extending it with zeroes we obtain an optimal integral solution (y^*, y^*_g, y^*_f) of (4)-(6).

If $\mathcal{F}_{y'}$ is not cross-free then we use the following uncrossing process. We take crossing sets $Z, S \in \mathcal{F}_{y'}$ and we decrease the values $y'(Z), y'(S)$ by $\varepsilon := \min\{y'(Z), y'(S)\}$ and we increase the values $y'(Z \cup Y)$ and $y'(Z \cap Y)$ by ε. We claim that the new y'' that we obtained together with the unchanged y'_g, y'_f is still an optimal solution of the dual. Indeed, because of the crossing submodularity of b we do not increase the quantity at (7) and it is easy to check that (8) and (9) remain true. To show that after finitely many iteration the support of the new rational optimal solution is cross-free, we need the following Lemma.

3.8 Lemma. *Let $a_1 \dots, a_n \in \mathbb{Q}^+$. Consider the following operation. Take $1 \le i < j < k < l \le n$ where $\min\{a_j, a_k\} = \varepsilon > 0$ (if it is possible). Then decrease a_j, a_k by ε and increase a_i, a_l by ε. Iterate this operation with the modified a_i. We claim that the process stops after finitely many steps.*

Proof. By multiplying every a_i with the same integer, we can assume that $a_i \in \mathbb{Z}$. We use induction on n. For $n < 4$, one cannot apply the operation at all. Suppose that we know for some k that the process terminates with any initial k-tuple. Let $n = k + 1$ and consider a_1. It can only increase during the iteration but $\sum_{i=1}^{k+1} a_i$ remains constant thus after at most $\sum_{i=2}^{k+1} a_i$ increases a_1 changes no more. In the remaining part of the process we have only k members thus it is finite. □

Let $\mathcal{F} = \{Z_1, \dots, Z_n\}$ where $Z_i \subseteq Z_j$ for $i \le j$ and let $a_i := y'(Z_i)$. Lemma 3.8 ensures that the uncrossing process terminates after finitely many steps. □

3.9 Corollary. *If in Theorem 3.7 we have a lower bound $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ which is crossing supermodular (instead of the upper bound b), then the system*

$$\begin{aligned} p &\le Qx \\ f \le x &\le g \end{aligned}$$

is TDI if it is solvable.

Proof. Consider Theorem 3.7 and take $-p$ as b and reverse the edges in the digraph. Then in the primal we have $(-Q)x \le -p$ i.e., $Qx \ge p$. □

3.10 Remark. If there is no lower or upper bounds for some $x(e)$ at Theorem 3.7, then it is still TDI. (The proof is essentially the same, just the notation is more complicated). It means we can allow $f : A \to \mathbb{Z} \cup \{-\infty\}$ and $g : A \to \mathbb{Z} \cup \{+\infty\}$ at Theorem 3.7.

3.11 Theorem. *Consider the polyhedron at Theorem 3.7. It is nonempty if and only if $\varrho_f(Z) - \delta_g(Z) \le b(Z)$ for every $Z \subseteq V$.*

Applications of subflows

3.12 Theorem (Nash-Williams). *A graph $G = (V, E)$ has a k-edge-connected orientation if and only if G is $2k$-edge-connected.*

Proof. To prove the nontrivial direction, fix first and arbitrary orientation D of G. We are looking for an $x^* \in \{0, 1\}^A$ such that by reversing the edges $\{e \in A : x^*(e) = 1\}$ in D we obtain a k-edge-connected orientation. Consider the system

$$\begin{aligned} &\underline{0} \le x \le \underline{1} \\ &\varrho_D(Z) - \varrho_x(Z) + \delta_x(Z) \ge k \quad (\varnothing \subsetneq Z \subsetneq V). \end{aligned}$$

If it has an integral solution then that is appropriate choice for x^*. We have at least a fractional solution namely $x \equiv \frac{1}{2}$. Transform the second inequality to the form $\varrho_x(Z) - \delta_x(Z) \le \varrho_D(Z) - k$. Let

$$b(Z) := \begin{cases} \varrho_D(Z) - k & \text{if } \varnothing \subsetneq Z \subsetneq V \\ 0 & \text{if } Z \in \{\varnothing, V\}. \end{cases}$$

It is routine to check that ϱ_D is submodular and hence $\varrho_D(Z) - k$ as well. It implies that b is crossing submodular. By Theorem 3.7, the system is TDI hence the desired integral solution exists by Theorem 3.3. □

Let $D = (V, A)$ be a digraph. A nonempty $F \subseteq A$ is called a directed cut (shortly **dicut**) if there is a bipartition (S, T) of V such that every edge between S and T points towards T and F consists of exactly these edges. A **dijoin** of D is a $J \subseteq A$ that meets all the dicuts.

3.13 Theorem (Lucchesi-Younger)**.** *In any digraph $D = (V, A)$, the minimal size of a dijoin equals to the maximal number of disjoint dicuts.*

Proof. We can assume that D is weakly connected. Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ such that

$$p(Z) = \begin{cases} 1 & \text{if } \delta_D(Z) = 0 \text{ where } Z \neq \varnothing, V \\ -\infty & \text{otherwise.} \end{cases}$$

Furthermore, let $f \equiv 0$, $g \equiv +\infty$ and $c \equiv 1$ on A. It is routine to check that p is crossing supermodular. The subflow polyhedron defined by D, f, g, p is nonempty since $x \equiv 1$ is in it. Consider the following primal and dual programs.

$$\begin{aligned} &\min \underline{1}x \\ &x \geq \underline{0} \\ &Qx \geq \underline{1} \end{aligned}$$

$$\begin{aligned} &\max y\underline{1} \\ &y \geq \underline{0} \\ &yQ \leq \underline{1} \end{aligned}$$

It is TDI by Corollary 3.9. It follows from Theorem 3.3 that there is an integral optimal primal x^* and integral optimal dual solution y^*. It means that x^* is the characteristic function of a dijoin and $\{\mathsf{in}_D(Z) : y^*(Z) = 1\}$ is a family of disjoint dicuts. By Strong Duality 1.25, $\underline{1}x^* = y^*\underline{1}$ thus the dijoin and the dicut family we have just found are of the same size. $\square$

3.14 Theorem. *Let $b^*, p^* : 2^V \to \mathbb{Z}$ where b^* is submodular, p^* is supermodular and $p^* \leq b^*$. Then there is a modular $m : 2^V \to \mathbb{Z}$ such that $p^* \leq m \leq b^*$.*

Proof. By subtracting the same constant function from b^* and p^*, we may assume that $p^*(\varnothing) = 0$. We can assume that $b^*(\varnothing) = 0$ as well since changing this value to zero does not ruin submodularity. Take two disjoint copies V' and V'' of V. For every $v \in V$ let $v'v''$ be an edge with $f(v'v'') := -\infty$ and $g(v'v'') := +\infty$. Finally $b(Z) := b^*(Z \cap (V' \cup V'')) - p^*(Z \cap (V' \cup V''))$.

It is not too hard to check that b is submodular and the condition in Theorem 3.11 hold. Hence there is a subflow x^* which we can assume is integral by Theorem 3.7. Then $m(Z) := \sum_{v \in Z} x^*(v'v'')$ satisfies the conditions. $\square$